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Maximal left regular submonoids and right regular submonoids of $Hyp_G(n)^*$

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Abstract. A generalized hypersubstitution of type $\tau = (n)$ is a mapping which maps each $n-$ary operation symbol to a term which does not necessarily preserve the arity. For every generalized hypersubstitution can be extended to a mapping defined on the set of all terms of type $\tau = (n)$. Then we can define a binary operation on the set of all generalized hypersubstitutions of type $\tau = (n)$ and it turns out that the set together with this binary operation forms a monoid. The concepts of left regular and right regular elements are important role in semigroup theory. In this paper, we characterize the set of all left regular and the set of all right regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (n)$ and we determine all maximal left regular submoniods and all maximal right regular submoniods of this monoid.

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1. Introduction and Preliminaries

The notions of hyperidentities and hypervarieties of a given type τ without nullary operations originated by J.Aczèl [1], V.D. Belousov [2], W.D. Neumann [9] and W. Taylor [16]. The main tool used to study hyperidentities and hypervarieties is the concept of a hypersubstitution which was introduced by W. Taylor [16]. The notation of a hypersubstution was originated by K. Denecke, D. Lau, R. Pöschel and D. Schweigert [6].

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In 2000, S. Leeratanavalee and K. Denecke generalized the concepts of a hypersubstitution and a hyperidentity to the concepts of a generalized hypersubstitution and a strong hyperidentity, respectively [8]. The set of all generalized hypersubstitutions together with a binary operation forms a monoid.

All regular elements of the monoid of all generalized hypersustitutions of type $\tau = (2)$ was studied by W. Puninagool and S. Leeratanavalee [11]. All idempotent and regular elements of the monoid of all generalized hypersustitutions of type $\tau = (3)$ was studied by S. Sudsanit and S. Leeratanavalee [11].

In 2010, W. Puninagool and S. Leeratanavalee characterized all idempotent and regular elements of the monoid of all generalized hypersustitutions of type $\tau = (n)$ [13]. In 2014, S. Sudsanit, S. Leeratanavalee and W. Puninagool characterized left-right regular elements of the monoid of all generalized hypersustitutions of type $\tau = (2)$ [13]. The set of all completely regular elements of this monoid of type $\tau = (n)$ was determined by A. Boonmee and S. Leeratanavalee [4]. In general, a completely regular element is both left regular and right regular.

In the present paper, we used the concepts of a regular element and a completely regular element as tools to determine the set of all left regular and right regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (n)$. Furthermore, we show that the set of all completely regular elements, the set of all left regular and the set of all right regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (n)$ are the same. Finally, we determine all maximal left regular submonoids and all maximal right regular submonoids of this monoid.

One of the most important tools in the study of universal algebra and theoretical computer science is the notion of terms. Let $\tau = (n_i)_{i\in I}, n_i \in \mathbb{N}$, be a type with operation symbols f_i for each $i \in I$. Let $X := \{x_1, x_2, ...\}$ be a countably infinite set of variables and $X_n := \{x_1, x_2, ..., x_n\}$ be an $n-$element alphabet. An $n-$ary term of type τ, for simply an $n-$ary term, is defined inductively as follows:

(i) The variables $x_1, x_2, ..., x_n$ are $n-$ary terms.

(ii) If $t_1, t_2, ..., t_{n_i}$ are $n-$ary terms of type τ then $f_i(t_1, t_2, ..., t_{n_i})$ is an $n-$ary term.

Let $W_\tau(X_n)$ be the set of all $n-$ary terms. $W_\tau(X_n)$ is the smallest set which contains $x_1, x_2, ..., x_n$ and is closed under finite application of (ii). Let $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ and called the set of all terms of type τ.

A generalized hypersubstitution of type τ is a mapping $\sigma : \{f_i | i \in I\} \to W_\tau(X)$ which does not necessarily preserve the arity. The set of all generalized hypersubstitutions of type τ denoted by $Hyp_G(\tau)$. To define a binary operation on this set, we need the concept of a generalized superposition of terms $S^m : W_\tau(X)^{m+1} \to W_\tau(X)$ which is defined by the following steps:

(i) If $t = x_j$, $1 \leq j \leq m$, then $S^m(t, t_1, ..., t_m) = S^m(x_j, t_1, ..., t_m) := t_j$.

(ii) If $t = x_j$, $m < j \in \mathbb{N}$, then $S^m(t, t_1, ..., t_m) = S^m(x_j, t_1, ..., t_m) := x_j$.

(iii) If $t = f_i(s_1, s_2, ..., s_{n_i})$, then
$S^m(t, t_1, ..., t_m) := f_i(S^m(s_1, t_1, ..., t_m), ..., S^m(s_{n_i}, t_1, ..., t_m))$.

For each generalized hypersubstitution σ can be extended to a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ defined as follows:

(i) $\hat{\sigma}[x] := x \in X$,

(ii)

$$\hat{\sigma}[f_i(t_1, t_2, ..., t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], ..., \hat{\sigma}[t_{n_i}]),$$

for any n_i-ary operation symbol f_i and supposed that $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

A binary operation $\circ_G$ on $Hyp_G(\tau)$ defined by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings. In [8], S. Leeratanavalee and K. Denecke showed that the set of all generalized hypersubstituions forms a monoid under the operation $\circ_G$ where the identity σ_{id} is a generalized hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, x_2, ..., x_{n_i})$.

2. Main results

At first, we introduce some notations which will be used throughout of this paper. Let $\tau = (n)$ be a type with an $n-$ary operation symbol f and let $t \in W_\tau(X)$, we denote

σ_t := the generalized hypersubstitution σ of type $\tau = (n)$ which maps f to the term t,

$var(t)$:= the set of all variables occurring in the term t,

$vb^t(x)$:= the number of occurrences of a variable x in the term t,

$op(t)$:= the number of all operation symbols occurring in the term t.

For a term $t \in W_{(n)}(X)$, a subterm of t is defined inductively by the following:

(i) Every variable $x \in var(t)$ is a subterm of t.

(ii) If $t = f(t_1, ..., t_n)$, then $t_1, ..., t_n$ and t itself are subterms of t.

We denote the set of all subterms of t by $sub(t)$.

For each $t \in W_{(n)}(X) \setminus X$ where $t = f(t_1, ..., t_n)$ for some $t_1, ..., t_n \in$

$W_{(n)}(X)$. Let $\pi_{i_l} : W_{(n)}(X) \setminus X \to W_{(n)}(X)$ with $\pi_{i_l}(t) = \pi_{i_l}(f(t_1, ..., t_n)) = t_{i_l}$. Maps π_{i_l} are defined for $i_l = 1, 2, ..., n$. Let $s^{(j)}$ be a subterm s occurring in the j^{th} order of t (from the left). If $s^{(j)} = \pi_{i_m} \circ ... \circ \pi_{i_1}(t)$ for some $m \in \mathbb{N}$, then the sequence of $s^{(j)}$ in t denoted by $seq^t(s^{(j)})$ and the depth of $s^{(j)}$ in t denoted by $depth^t(s^{(j)})$ such that

$$seq^t(s^{(j)}) = (i_1, ..., i_m) \quad \text{and} \quad depth^t(s^{(j)}) = m.$$

Let $s \in sub(t)$ where $s \neq t$. We denote the set of all a sequences of s in term t by $seq^t(s)$, then

$$seq^t(s) = \{seq^t(s^{(j)}) | \, j \in \mathbb{N}\}.$$

Example 2.1. Let $t \in W_{(4)}(X) \setminus X$ where

$$t = f(x_4, f(s, f(x_4, s, s, x_3), x_1, x_5), s, x_5)$$

for some $s \in W_{(4)}(X)$. Then

$$t = f(x_4, f(s^{(1)}, f(x_4, s^{(2)}, s^{(3)}, x_3), x_1, x_5), s^{(4)}, x_5)$$

and then

$$\begin{aligned} seq^t(s^{(1)}) &= (2,1) \qquad & depth^t(s^{(1)}) &= 2 \\ seq^t(s^{(2)}) &= (2,2,2) \qquad & depth^t(s^{(2)}) &= 3 \\ seq^t(s^{(3)}) &= (2,2,3) \qquad & depth^t(s^{(3)}) &= 3 \\ seq^t(s^{(4)}) &= (3) \qquad & depth^t(s^{(4)}) &= 1 \end{aligned}$$

and $seq^t(s) = \{(2,1), (2,2,2), (2,2,3), (3)\}$.

Definition 2.2 ([11]). Let S be a semigroup and a be an element in S. Then

a is called *regular* iff there exists $b \in S$ such that $aba = a$,

a is called *completely regular* iff there exists $b \in S$ such that $aba = a$ and $ab = ba$.

Let $\sigma_t \in Hyp_G(n)$, we denote

$R_1 := \{\sigma_{x_i} | x_i \in X\}$;

$R_2 := \{\sigma_t | var(t) \cap X_n = \emptyset\}$;

$R_3 := \{\sigma_t | t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{j_1}, ..., t_{i_m} = x_{j_m}$ for some $i_1, \ldots, i_m$ and for distinct $j_1, \ldots, j_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{j_1}, \cdots, x_{j_m}\}\}$.

In 2010, W. Puninagool and S. Leeratanavalee showed that $\bigcup_{i=1}^{3} R_i$ is the set of all regular elements in $Hyp_G(n)$ [13].

Denote

$CR(R_3) := \{\sigma_t | t \in W_{(n)}(X) \setminus X$ such that $t = f(t_1, ..., t_n)$ then there exist $t_{i_1}, \ldots, t_{i_m} \in \{t_1, \ldots, t_n\}$ such that $t_{i_1} = x_{\pi(i_1)}, ..., t_{i_m} = x_{\pi(i_m)}$ where π is a bijective map on $\{i_1, ..., i_m\}$ and $var(t) \cap X_n = \{x_{i_1}, ..., x_{i_m}\}\}$.

Clearly, $CR(R_3) \subset R_3$. Let $CR(Hyp_G(n)) := CR(R_3) \cup R_1 \cup R_2$. In 2013, A. Boonmee and S. Leeratanavalee showed that $CR(Hyp_G(n))$ is the set of all completely regular elements in $Hyp_G(n)$ [4].

Let $t \in W_{(n)}(X)$ and $i \in \mathbb{N}$ which $1 \leq i \leq n$, an $i - most(t)$ is defined inductively by the following :

(i) If t is a variable, then $i - most(t) = t$.

(ii) If $t = f(t_1, \ldots, t_n)$, then $i - most(t) = i - most(t_i)$.

Let $\sigma_t \in Hyp_G(n)$ and let $\emptyset \neq I \subset \{1, \ldots, n\}$. Denote,

$CR_1(R_3) := \{\sigma_t | t = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ where π is a bijective map on $\{1, \ldots, n\}\}$,

$E := \{\sigma_t | t = f(t_1, \ldots, t_n)$ where $t_{i_1} = x_{i_1}, \ldots, t_{i_m} = x_{i_m}$ for some $t_{i_1}, \ldots, t_{i_m} \in \{t_1, \ldots, t_n\}$ and $var(t) \cap X_n = \{x_{i_1}, \ldots, x_{i_m}\}$ and if $x_{i_l} \in$

$var(t_k)$ for some $i_l \in \{i_1, \ldots, i_m\}$ and $k \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_m\}$, then $j - most(t_k) \neq x_{i_l}$ for all $j \neq i_l\}$,

$CR_I(R_3) := \{\sigma_t | t = f(t_1, \ldots, t_n)$ where $t_i = x_{\pi(i)}$ for all $i \in I$ and π is a bijective map on I, $var(t) \cap X_n = \{x_{\pi(i)} | i \in I\}\}$,

$CR'_I(R_3) := \{\sigma_t | t = f(t_1, \ldots, t_n)$ where $t_i = x_{\pi(i)}$; $\pi(i) \in I$ for all $i \in I$ and $t_k = x_{\pi(k)}$ for all $k \in \{1, \ldots, n\} \setminus I$ and π is a bijective map on $I\}$ and denote,

$(MCR)_{Hyp_G(n)} := R_1 \cup R_2 \cup CR_1(R_3)$,

$(MCR_1)_{Hyp_G(n)} := R_1 \cup R_2 \cup E$,

$(MCR_I)_{Hyp_G(n)} := R_1 \cup R_2 \cup CR_I(R_3) \cup CR'_I(R_3) \cup \{\sigma_{id}\}$.

In 2019, P. Kunama and S. Leeratanavalee [7] showed that $(MCR)_{Hyp_G(n)}$ and $(MCR_1)_{Hyp_G(n)} \cup (MCR_I)_{Hyp_G(n)}$ are all maximal completely regular submonoids of $Hyp_G(n)$.

Defintion 2.3 [10]. Let S be a semigroup and a be an element in S. Then a is called *left(right) regular* iff $a \in Sa^2$ $(a \in a^2S)$.

Theorem 2.4 [10]. *An element a of a semigroup S is completely regular if and only if a is both left regular and right regular.*

Proposition 2.5 [4]. *Let $t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{j_1}, ..., t_{i_m} = x_{j_m}$ for some $i_1, \ldots, i_m$, $j_1, \ldots, j_m \in \{1, ..., n\}$. If there exists $l \in \{1, ..., m\}$ such that $t_{i_l} = x_{j_l}$ where $j_l \notin \{i_1, ..., i_m\}$, then $\sigma_t \neq \sigma_s \circ_G \sigma_t^2$ for all $\sigma_s \in Hyp_G(n)$.*

Corollary 2.6 [4]. *If $\sigma_t \in R_3 \setminus CR(R_3)$, then σ_t is not left regular in $Hyp_G(n)$.*

Proposition 2.7. *Let $t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{j_1}, ..., t_{i_m} = x_{j_m}$ for some $i_1, \ldots, i_m$, $j_1, \ldots, j_m \in \{1, ..., n\}$. If there exists $l \in \{1, ..., m\}$ such* *that* $t_{i_l} = x_{j_l}$ *where* $j_l \notin \{i_1, ..., i_m\}$, *then* $\sigma_t \neq \sigma_t^2 \circ_G \sigma_s$ *for all* $\sigma_s \in Hyp_G(n)$.

Proof. Assume that the condition holds. Consider

$$(\sigma_t \circ_G \sigma_t)(f) = \hat{\sigma}_t[t] = S^n(f(t_1, ..., t_n), \hat{\sigma}_t[t_1], ..., \hat{\sigma}_t[t_n]) = f(u_1, ..., u_n)$$

where $u_i = S^n(t_i, \hat{\sigma}_t[t_1], ..., \hat{\sigma}_t[t_n])$ for all $i \in \{1, ..., n\}$, denote $(\sigma_t \circ_G \sigma_t)(f) = u$. Since $t_{i_l} = x_{j_l}$ where $j_l \notin \{i_1, \cdots, i_m\}$, $u_{i_l} = S^n(x_{j_l}, \hat{\sigma}_t[t_1], \cdots, \hat{\sigma}_t[t_n]) = \hat{\sigma}_t[t_{j_l}]$. So $u_{i_l} \in W_{(n)}(X) \setminus X_n$.

Let $\sigma_t \in Hyp_G(n)$. Next, we will show that $\sigma_t \neq \sigma_t^2 \circ_G \sigma_s$. If $s \in X$, then $\sigma_t^2 \circ_G \sigma_s \in X$. So $\sigma_t \neq \sigma_t^2 \circ_G \sigma_s$. If $s = f(s_1, s_2, ..., s_n)$ where $s_1, s_2, ..., s_n \in W_{(n)}(X)$, then

$$\begin{aligned}(\sigma_t^2 \circ_G \sigma_s)(f) &= (\sigma_u \circ_G \sigma_s)(f) \\ &= S^n(f(u_1, ..., u_n), \hat{\sigma}_u[s_1], ..., \hat{\sigma}_u[s_n]) \\ &= f(w_1, ..., w_n)\end{aligned}$$

where $w_i = S^n(u_i, \hat{\sigma}_u[s_1], ..., \hat{\sigma}_u[s_n])$ for all $i \in \{1, ..., n\}$. Since $u_{i_l} \in W_{(n)}(X) \setminus X_n$, $w_{i_l} \in W_{(n)}(X) \setminus X_n$. Hence $\sigma_t \neq \sigma_t^2 \circ_G \sigma_s$. □

Corollary 2.8. *If* $\sigma_t \in R_3 \setminus CR(R_3)$, *then* σ_t *is not right regular in* $Hyp_G(n)$.

Proposition 2.9 [12]. *Let* $s, t_1, ..., t_m \in W_\tau(X)$. *Then*

$$op(S^m(s, t_1, ..., t_m)) = \sum_{j=1}^{m} vb^s(x_j)op(t_j) + op(s)$$

Theorem 2.10. *If* $\sigma_t \in Hyp_G(n) \setminus (R_1 \cup R_2 \cup R_3)$, *then* σ_t *is not right regular in* $Hyp_G(n)$.

Proof. Let $\sigma_t \in Hyp_G(n) \setminus (R_1 \cup R_2 \cup R_3)$ where $t = f(t_1, ..., t_n)$.

Denote

$$\begin{aligned} I_1 &= \{i \in \{1,...,n\} | t_i \in X \setminus X_n\}, \\ I_2 &= \{i \in \{1,...,n\} | t_i \in X_n\}, \\ I_3 &= \{i \in \{1,...,n\} | t_i \in W_{(n)}(X) \setminus X\}. \end{aligned}$$

Clearly, I_1, I_2 and I_3 are all distinct and $I_1 \cup I_2 \cup I_3 = \{1,...,n\}$. Then there exists $x_j \in var(t)$ for some $j \in \{1,...,n\}$ where $x_j \notin \{t_i | i \in I_2\}$.

Suppose that $\sigma_t = \sigma_t^2 \circ_G \sigma_s$ for some $\sigma_s \in Hyp_G(n)$. By Proposition 2.7, $j \notin I_2$. So $j \in I_3$. Then there exists $t_j \in W_{(n)}(X) \setminus X$ such that $op(\hat{\sigma}_t[t_j]) \geq op(t)$. Consider

$$\begin{aligned} (\sigma_t \circ_G \sigma_t)(f) &= S^n(f(t_1,...,t_n), \hat{\sigma}_t[t_1],...,\hat{\sigma}_t[t_n]) \\ &= f(u_1,...,u_n), \end{aligned}$$

where $u_i = S^n(t_i, \hat{\sigma}_t[t_1],...,\hat{\sigma}_t[t_n])$ for all $i \in \{1,...,n\}$. If $i \in I_1$, then $u_i = t_i$ such that $u_i \in X \setminus X_n$. If $i \in I_2$, then $u_i \in \{t_i | i \in I_2\}$, by Proposition 2.7. If $i \in I_3$, then $u_i \in W_{(n)}(X) \setminus X$. Choose $k \in I_3$ where $x_j \in var(t_k)$. Then $op(t_k) \geq 1$, $u_k \in W_{(n)}(X) \setminus X$ and $vb^{t_k}(x_j) op(\hat{\sigma}_t[t_j]) \geq op(t)$. So

$$\begin{aligned} op(u_k) &= \sum_{p=1}^{n} vb^{t_k}(x_p) op(\hat{\sigma}_t[t_p]) + op(t_k) \\ &\geq op(t) + op(t_k) \\ &\geq op(t) + 1 \\ &> op(t). \end{aligned}$$

Hence $op(\sigma_t^2) > op(t)$, i.e. $op(\sigma_t^2 \circ_G \sigma_s) > op(t)$, which contradicts to $\sigma_t = \sigma_t^2 \circ_G \sigma_s$. Therefore σ_t is not a right regular element in $Hyp_G(n)$. Similarly, if $I_1 = \emptyset$ or $I_2 = \emptyset$, then σ_t is not a right regular element in $Hyp_G(n)$. □

Theorem 2.11 [5]. *Let $t, s \in W_{(n)}(X) \setminus X$ and $x_i \in var(t)$. Let $x_i^{(j)}$ be a variable x_i occurring in the j^{th} order of t (from the left) such that*

$seg^t(x_i^{(j)}) = (i_1, ..., i_m)$ for some $i_1, ..., i_m \in \{1, .., n\}$. Then $x_i^{(j)} \in var(\hat{\sigma}_s[t])$ if and only if $x_{i_k} \in var(s)$ for all $1 \leq k \leq m$. Let $x_i^{(j,h)}$ be a variable $x_i^{(j)}$ occurring in the h^{th} order of $\hat{\sigma}_s[t]$ (from the left), Then

$$seq^{\hat{\sigma}_s[t]}(x_i^{(j,h)}) = (a_{i_1}, ..., a_{i_m})$$

where a_{i_k} is a sequence of natural number $k_1, ...k_z$ such that $(k_1, ...k_z) \in seq^s(x_{i_k})$ for all $k \in \{1, .., m\}$. Moreover

$$depth^{\hat{\sigma}_s[t]}(x_i^{(j,h)}) = depth^s(x_{i_1}^{l_1}) + ... + depth^s(x_{i_m}^{l_m})$$

for some $l_1, ..., l_m \in \mathbb{N}$ where $x_{i_k}^{l_k}$ is a variable x_{i_k} occurring in the l_k^{th} order of S (from the left) for all $k \in \{1, ..., m\}$.

Theorem 2.12. *If $\sigma_t \in Hyp_G(n) \setminus (R_1 \cup R_2 \cup R_3)$, then σ_t is not left regular in $Hyp_G(n)$.*

Proof. Let $\sigma_t \in Hyp_G(n) \setminus (R_1 \cup R_2 \cup R_3)$ where $t = f(t_1, ..., t_n)$. Denote I_1, I_2 and I_3 as in Theorem 2.10.

Suppose that $\sigma_t = \sigma_s \circ_G \sigma_t^2$ for some $\sigma_s \in Hyp_G(n)$. By Proposition 2.5, $j \in I_3$. Since $x_j \in var(t)$ and $\sigma_t = \sigma_s \circ_G \sigma_t^2$, $x_j \in var(\sigma_s \circ_G \sigma_t^2)$, i.e. $x_j \in var(\sigma_t^2)$. Let $x_j^{(h)}$ be a variable x_j occurring in the h^{th} order of t (from the left) where

$$\begin{aligned} depth^t(x_j^{(h)}) &= min\{depth^t(x_j^{(z)}) | x_j^{(z)} \text{ is a variable } x_j \text{ occurring in the } z^{th} \\ &\quad \text{order of } t \text{ (from the left)}\} \\ &= m. \end{aligned}$$

Then $seq^t(x_j^{(h)}) = (k_1, k_2, ..., k_m)$ where $k_1 \in I_3$ and $k_2, ..., k_m \in \{1, 2, ..., n\}$ and we get $x_j^{(h)} \in var(\sigma_t^2)$. Fix $x_j^{(h,p)}$ is a variable $x_j^{(h)}$ occurring in the p^{th}

order of σ_t^2 (from the left) where

$$\begin{aligned} depth^{\sigma_t^2}(x_j^{(h,p)}) &= min\{depth^{\sigma_t^2}(x_j^{(h,q)})|x_j^{(h,q)} \text{ is a variable } x_j^{(h)} \text{ occurring} \\ &\quad \text{in the } q^{th}\text{order of } \sigma_t^2 \text{ (from the left)}\} \\ &= min\{depth^{\sigma_t^2}(x_j^{(s)})|x_j^{(s)} \text{ is a variable } x_j \text{ occurring in the} \\ &\quad s^{th}\text{order of } \sigma_t^2 \text{ (from the left)}\}. \end{aligned}$$

By Theorem 2.11, we have $x_{k_1}, x_{k_2}, ..., x_{k_m} \in var(t)$. By Proposition 2.5 and $k_1 \notin I_2$, we have $x_{k_1} \notin \{t_i|i \in I_2\}$. So $x_{k_1} \in var(t_l)$ for some $l \in I_3$, i.e. $depth^t(x_{k_1}^r) \geq 2$ for all $r \in \mathbb{N}$ such that $x_{k_1}^r$ is a variable x_{k_1} occurring in the r^{th} order of t (from the left). By Theorem 2.11,

$$\begin{aligned} depth^{\sigma_t^2}(x_j^{(h,p)}) &= depth^t(x_{k_1}^{r_1}) + depth^t(x_{k_2}^{r_2}) + ... + depth^t(x_{k_m}^{r_m}) \\ &\geq 2 + (m-1) \\ &> m \end{aligned}$$

where $depth^t(x_{k_i}^{r_i})$ is a variable x_{k_i} occurring in the r_i^{th} order of t (from the left) for all $i \in \{1, ..., m\}$. Hence

$$\begin{aligned} m &< depth^{\sigma_t^2}(x_j^{(h,p)}) \\ &\leq min\{depth^{\sigma_s \circ_G \sigma_t^2}(x_j^{(s)})|x_j^{(s)} \text{ is a variable } x_j \text{ occurring in the } s^{th} \\ &\quad \text{order of } \sigma_s \circ_G \sigma_t^2 \text{ (from the left)}\}, \end{aligned}$$

which contradicts to $\sigma_t = \sigma_s \circ_G \sigma_t^2$. Therefore σ_t is not a left regular element in $Hyp_G(n)$. Similarly, if $I_1 = \emptyset$ or $I_2 = \emptyset$, then σ_t is not a left regular element in $Hyp_G(n)$. □

Theorem 2.13. *$CR(Hyp_G(n))$ is the set of all left regular elements in $Hyp_G(n)$.*

Proof. By Corollary 2.6 and by Theorem 2.12. □

Theorem 2.14. *$CR(Hyp_G(n))$ is the set of all right regular elements in $Hyp_G(n)$.*

Proof. By Corollary 2.8 and by Theorem 2.10. □

Theorem 2.15. *Let $\emptyset \neq I \subset \{1, \ldots, n\}$. Then $(MCR)_{Hyp_G(n)}$ and $(MCR_1)_{Hyp_G(n)} \cup (MCR_I)_{Hyp_G(n)}$ are all maximal left regular and all maximal right regular submonoids of $Hyp_G(n)$.*

Proof. By Theorem 2.13, Theorem 2.14 and[7]. □

3. Conclusion

In this paper, we conclude that the set of all completely regular elements, the set of all left regular elements and the set of all right regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (n)$ are the same. In semigroup theory, we know that the set of all completely regular elements is a subset of the set all intra-regular elements. A. Boonmee [3] showed that the set of all completely regular elements and the set of all intra-regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (n)$ are the same.

It follows that, completely regular, left regular, right regular and intra-regular of the monoid of all generalized hypersubstitutions of type $\tau = (n)$ are all equivalent. Moreover, we have $(MCR)_{Hyp_G(n)}$ and $(MCR_1)_{Hyp_G(n)} \cup (MCR_I)_{Hyp_G(n)}$ where $\emptyset \neq I \subset \{1, ..., n\}$ are all maximal completely regular (left regular, right regular and intra-regular) submonoids of the monoids of all generalized hypersubstitutions of type $\tau = (n)$.

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